

ON p -ADIC HURWITZ-TYPE EULER ZETA FUNCTIONS

MIN-SOO KIM AND SU HU

ABSTRACT. Henri Cohen and Eduardo Friedman constructed the p -adic analogue for Hurwitz zeta functions, and Raabe-type formulas for the p -adic gamma and zeta functions from Volkenborn integrals satisfying the modified difference equation. In this paper, we define the p -adic Hurwitz-type Euler zeta functions. Our main tool is the fermionic p -adic integral on \mathbb{Z}_p . We find that many interesting properties for the p -adic Hurwitz zeta functions are also hold for the p -adic Hurwitz-type Euler zeta functions, including the convergent Laurent series expansion, the distribution formula, the functional equation, the reflection formula, the derivative formula, the p -adic Raabe formula and so on.

1. INTRODUCTION

Throughout this paper, \mathbb{C} the field of complex numbers, p will denote an odd rational prime number, \mathbb{Z}_p the ring of p -adic integers, \mathbb{Q}_p the field of fractions of \mathbb{Z}_p , $\mathbb{C}_p = \widehat{\overline{\mathbb{Q}_p}}$ the Tate field (the completion of a fixed algebraic closure $\overline{\mathbb{Q}_p}$ of \mathbb{Q}) and $C\mathbb{Z}_p = \mathbb{Q}_p \setminus \mathbb{Z}_p$. Let v_p be the p -adic valuation of \mathbb{C}_p normalized so that $|p|_p = p^{-v_p(p)} = p^{-1}$.

We say that $f : \mathbb{Z}_p \rightarrow \mathbb{C}_p$ is uniformly differentiable function at a point $a \in \mathbb{Z}_p$, and we write $f \in UD(\mathbb{Z}_p)$, if the difference quotients $\Phi_f : \mathbb{Z}_p \times \mathbb{Z}_p \rightarrow \mathbb{C}_p$ such that

$$(1.1) \quad \Phi_f(x, y) = \frac{f(x) - f(y)}{x - y}$$

have a limit $f'(a)$ as $(x, y) \rightarrow (a, a)$, $x \neq y$ (see [20, p. 221]). The fermionic p -adic integral $I_{-1}(f)$ on \mathbb{Z}_p is defined by

$$(1.2) \quad I_{-1}(f) = \int_{\mathbb{Z}_p} f(a) d\mu_{-1}(a) = \lim_{N \rightarrow \infty} \sum_{a=0}^{p^N-1} f(a) (-1)^a.$$

where $f \in UD(\mathbb{Z}_p)$. The fermionic p -adic integral $I_{-1}(f)$ on \mathbb{Z}_p is used by Taekyun Kim [13] to derives useful formulas involving the Euler numbers and polynomials, and it has also been used by the first author to give a brief proof of Stein's classical result on Euler numbers modulo power of two (see [10]).

Henri Cohen and Eduardo Friedman [5] constructed the p -adic analogue for Hurwitz zeta functions, and Raabe-type formulas for the p -adic gamma

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and zeta functions from Volkenborn integrals satisfying the modified difference equation. Here the Volkenborn integral of a function $f : \mathbb{Z}_p \rightarrow \mathbb{C}_p$ with $f \in UD(\mathbb{Z}_p)$ is defined by

$$(1.3) \quad \int_{\mathbb{Z}_p} f(x) dx = \lim_{N \rightarrow \infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} f(x)$$

(cf. [20, p. 264]). This integral was introduced by Volkenborn [22] and he also investigated many important properties of p -adic valued functions defined on the p -adic domain (see [22, 23]).

As shown by Cohen, the p -adic functions with nice properties are powerful tools for studying many results of classical number theory in a straightforward manner, for instance strengthenings of almost all the arithmetic results on Bernoulli and Euler numbers, see Cohen [4, Chapter 11] or the accounts in Iwasawa [7], Koblitz [15], Lang [17], Ram Murty [19], Washington [25].

In [4, Chapter 11], Cohen first gave the definition for the p -adic Hurwitz zeta functions for $x \in C\mathbb{Z}_p$, then he gave the definition for the p -adic Hurwitz zeta functions for $x \in \mathbb{Z}_p$. Many interesting properties for the p -adic Hurwitz zeta functions and L -functions have also been given in this chapter.

For $s \in \mathbb{C}$ and $\text{Re}(s) > 0$, the Euler zeta function and the Hurwitz-type Euler zeta function are defined by

$$\zeta_E(s) = 2 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^s}, \text{ and } \zeta_E(s, x) = 2 \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+x)^s}$$

respectively (see [1, 18, 9, 14]). Notice that the Euler zeta functions can be analytically continued to the whole complex plane, and these zeta functions have the values of the Euler number or the Euler polynomials at negative integers.

In this paper, we define the p -adic Hurwitz-type Euler zeta functions following Cohen's approach in [4, Chapter 11]. Our main tool is the fermionic p -adic integral $I_{-1}(f)$ on \mathbb{Z}_p . We find that many interesting properties for the p -adic Hurwitz zeta functions are also hold for the p -adic Hurwitz-type Euler zeta functions.

Our paper is organized as follows.

In section 2, we recall the relationship between the fermionic p -adic integral $I_{-1}(f)$ on \mathbb{Z}_p and the Euler numbers and polynomials.

In section 3, we gave the definition for the p -adic Hurwitz-type Euler zeta function for $x \in C\mathbb{Z}_p$. Many interesting properties for the p -adic Hurwitz-type Euler zeta function for $x \in C\mathbb{Z}_p$ have also been given, including the convergent Laurent series expansion, the distribution formula, the functional equation, the derivative formula, the p -adic Raabe formula and so on.

In section 4, we gave the definition for the p -adic Hurwitz-type Euler zeta function for $x \in \mathbb{Z}_p$ using characters modulo p^v . Many interesting properties for the p -adic Hurwitz-type Euler zeta function for $x \in \mathbb{Z}_p$ have also been given, including the distribution formula, the functional equation, the reflection formula, the derivative formula, the p -adic Raabe formula and

so on. By using the p -adic Hurwitz-type Euler zeta function, we also gave a new definition for the p -adic Euler ℓ -function for characters modulo p^v . We show that in this case the definition is equivalent to the first author's previous definition in [11]. In [11], the first author proposed a construction of p -adic Euler ℓ -function using Kubota-Leopoldt's approach and Washington's one and he also computed the derivative of p -adic Euler ℓ -function at $s = 0$ and the values of p -adic Euler ℓ -function at positive integers. Finally, we show the p -adic Hurwitz-type Euler zeta function can be represented as the p -adic Euler ℓ -functions using the power series expansion under certain conditions.

2. EULER NUMBERS AND POLYNOMIALS IN p -ADIC ANALYSIS

Let $X \subset \mathbb{C}_p$ be an arbitrary subset closed under $x \rightarrow x+a$ for $a \in \mathbb{Z}_p$ and $x \in X$. In particular, X could be $\mathbb{C}_p \setminus \mathbb{Z}_p$, $\mathbb{Q}_p \setminus \mathbb{Z}_p$ or \mathbb{Z}_p . Suppose $f : X \rightarrow \mathbb{C}_p$ is uniformly differentiable on X , so that for fixed $x \in X$ the function $a \rightarrow f(x+a)$ is uniformly differentiable on \mathbb{Z}_p . Let Δ be the difference operator defined by $(\Delta f)(a) = f(a+1) - f(a)$ and put $(\nabla f)(a) = f(a) - f(a-1)$. The following three properties of the fermionic p -adic integral $I_{-1}(f)$ on \mathbb{Z}_p can be directly derived from the definition:

$$(2.1) \quad \int_{\mathbb{Z}_p} f(x+a) d\mu_{-1}(a) = 2f(x-1) - \int_{\mathbb{Z}_p} (f_1)^{-1}(x+a) d\mu_{-1}(a);$$

$$(2.2) \quad \int_{\mathbb{Z}_p} f(x+a) d\mu_{-1}(a) = f(x) - \frac{1}{2} \int_{\mathbb{Z}_p} (\Delta f)(x+a) d\mu_{-1}(a);$$

$$(2.3) \quad \int_{\mathbb{Z}_p} f(x+a) d\mu_{-1}(a) = f(x-1) + \frac{1}{2} \int_{\mathbb{Z}_p} (\nabla f)(x+a) d\mu_{-1}(a).$$

In the p -adic theory (2.1)-(2.3) are the characterization integral equations of the Euler numbers and polynomials: If we put $f(a) = e^{at}$ in (2.1)-(2.3), we get

$$(2.4) \quad \int_{\mathbb{Z}_p} e^{at} d\mu_{-1}(a) + \int_{\mathbb{Z}_p} e^{(a-1)t} d\mu_{-1}(a) = 2e^{-t},$$

whence we may immediately deduce the following

$$(2.5) \quad e^{xt} \int_{\mathbb{Z}_p} e^{at} d\mu_{-1}(a) = \sum_{m=0}^{\infty} E_m(x) \frac{t^m}{m!},$$

where $t \in \mathbb{C}_p$ such that $|t|_p < p^{-1/(p-1)}$ and $E_m(x)$ are the Euler polynomials. Therefore by (2.5), we get

$$(2.6) \quad \int_{\mathbb{Z}_p} (x+a)^m d\mu_{-1}(a) = E_m(x).$$

If we put $f(a) = e^{(2a+1)t}$ with $t \in \mathbb{C}_p$ such that $|t|_p < p^{-1/(p-1)}$ in (2.1)-(2.3), we get

$$\int_{\mathbb{Z}_p} e^{(2(a+1)+1)t} d\mu_{-1}(a) + \int_{\mathbb{Z}_p} e^{(2a+1)t} d\mu_{-1}(a) = 2e^t,$$

whence we may immediately deduce the following

$$(2.7) \quad \int_{\mathbb{Z}_p} e^{(2a+1)t} d\mu_{-1}(a) = \sum_{m=0}^{\infty} E_m \frac{t^m}{m!},$$

where E_m are the Euler numbers (cf. [10]). By (2.6) and (2.7), we have the identity

$$E_m = \int_{\mathbb{Z}_p} (2a+1)^m d\mu_{-1}(a).$$

Using (2.6), this can also be written

$$(2.8) \quad \begin{aligned} E_m &= 2^m \int_{\mathbb{Z}_p} \left(a + \frac{1}{2}\right)^m d\mu_{-1}(a) \\ &= 2^m \sum_{k=0}^m \binom{m}{k} \left(\frac{1}{2}\right)^{m-k} E_k(0) \end{aligned}$$

and

$$(2.9) \quad \begin{aligned} E_m(0) &= 2^{-m} \int_{\mathbb{Z}_p} (2a+1-1)^m d\mu_{-1}(a) \\ &= 2^{-m} \sum_{k=0}^m \binom{m}{k} (-1)^{m-k} E_k. \end{aligned}$$

Taking $f(a) = e^{-(a+1)t}$ in (2.1)-(2.3), we find

$$(2.10) \quad \int_{\mathbb{Z}_p} e^{-(a+2)t} d\mu_{-1}(a) + \int_{\mathbb{Z}_p} e^{-(a+1)t} d\mu_{-1}(a) = 2e^{-t}.$$

It is easy to see from (2.4) that (2.10) satisfies

$$(2.11) \quad \int_{\mathbb{Z}_p} e^{-(a+1)t} d\mu_{-1}(a) = \frac{2e^{-t}}{e^{-t}+1} = \frac{2}{e^t+1} = \int_{\mathbb{Z}_p} e^{at} d\mu_{-1}(a).$$

Combining (2.5) and (2.11), we calculate

$$\begin{aligned} \sum_{m=0}^{\infty} \int_{\mathbb{Z}_p} (x+a)^m d\mu_{-1}(a) \frac{(-t)^m}{m!} &= e^{(1-x)t} \int_{\mathbb{Z}_p} e^{-(a+1)t} d\mu_{-1}(a) \\ &= e^{(1-x)t} \int_{\mathbb{Z}_p} e^{at} d\mu_{-1}(a) \\ &= \sum_{m=0}^{\infty} \int_{\mathbb{Z}_p} (1-x+a)^m d\mu_{-1}(a) \frac{t^m}{m!}, \end{aligned}$$

so that

$$(2.12) \quad (-1)^m \int_{\mathbb{Z}_p} (x+a)^m d\mu_{-1}(a) = \int_{\mathbb{Z}_p} (1-x+a)^m d\mu_{-1}(a), \quad m \geq 0,$$

which amounts to $E_m(1/2) = 0$ for odd m . From (2.8), we see that $E_{2m+1} = 0$ for $m \geq 0$.

3. THE p -ADIC HURWITZ-TYPE EULER ZETA FUNCTIONS

Before passing to the p -adic theory, we take a quick look at the complex analytic Hurwitz-type Euler zeta function $\zeta_E(s, x)$.

Definition 3.1. We define the Hurwitz-type Euler zeta function $\zeta_E(s, x)$ for $x \in \mathbb{R}_{>0}$ and $s \in \mathbb{C}$ with $\operatorname{Re}(s) > 0$ by

$$\zeta_E(s, x) = 2 \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+x)^s}.$$

It is known that $\zeta_E(s, x)$ can be extended to the whole s -plane by means of contour integral (cf. [14]).

For $x \in \mathbb{R}_{>0}$ and $m \geq 0$, the formula

$$(3.1) \quad \zeta_E(-m, x) = E_m(x)$$

holds (cf. [14]). This identity (3.1) obtained earlier in the course of Euler's proof for the values at non-positive integer arguments of the Riemann zeta function $\zeta(s) = \sum_{n=1}^{\infty} 1/n^s$ (cf. [1, 9]). From (2.6) and (3.1), we can write

$$(3.2) \quad \int_{\mathbb{Z}_p} (x+a)^m d\mu_{-1}(a) = E_m(x) = \zeta_E(-m, x).$$

This is used to construct a p -adic Hurwitz-type Euler zeta function.

Given $x \in \mathbb{Z}_p$, $p \nmid x$ and $p > 2$, there exists a unique $(p-1)$ th root of unity $\omega(x) \in \mathbb{Z}_p^\times$ such that

$$x \equiv \omega(x) \pmod{p},$$

where ω is the Teichmüller character. Let $\langle x \rangle = \omega^{-1}(x)x$, so $\langle x \rangle \equiv 1 \pmod{p}$. We extend the notation $\langle \cdot \rangle$ to \mathbb{Q}_p^\times by setting

$$(3.3) \quad \langle x \rangle = \left\langle \frac{x}{p^{v_p(x)}} \right\rangle.$$

If $x \in \mathbb{Q}_p^\times$, we define $\omega_v(x)$ by

$$(3.4) \quad \omega_v(x) = \frac{x}{\langle x \rangle} = p^{v_p(x)} \omega \left(\frac{x}{p^{v_p(x)}} \right)$$

(see [4, p. 280, Definition 11.2.2] for more details).

Definition 3.2. For $s \in \mathbb{C}_p$ such that $|s| \leq R_p = p^{(p-2)/(p-1)}$ and $x \in C\mathbb{Z}_p = \mathbb{Q}_p \setminus \mathbb{Z}_p$, we define the p -adic Hurwitz-type Euler zeta function $\zeta_{p,E}(s, x)$ by the equivalent formulas

$$\zeta_{p,E}(s, x) = \int_{\mathbb{Z}_p} \langle x+a \rangle^{1-s} d\mu_{-1}(a).$$

Remark 3.3. The theory of the p -adic analogue of the Riemann zeta function $\zeta(s) = \sum_n 1/n^s$ originates in the work of Kubota and Leopoldt [16], who used Kummer's congruences for Bernoulli numbers, generalized to special values of Hurwitz zeta-functions $\zeta(s, x) = \sum_n 1/(x+n)^s$ for $x \in \mathbb{R}_{>0}$. The function $\zeta_{p,E}(s, x)$ is a p -adic analogous of the Hurwitz-type Euler zeta function $\zeta_E(s, x)$ in this way. More accurately, the values are allowed to lie

in an algebraic closure of a p -adic field, for some given prime number p . Such functions can sometimes be defined by the p -adic interpolation of the values of the Hurwitz-type Euler zeta function $\zeta_E(s, x)$ at negative integers.

Lemma 3.4 ([19]). *Suppose that $r < p^{-1/(p-1)} < 1$ and for $s \in \mathbb{C}_p$*

$$f(s) = \sum_{i=0}^{\infty} \binom{s}{i} a_i$$

with $|a_i|_p \leq Mr^n$ for some M . Then $f(s)$ can be expressed as a power series with radius of convergence at least $R = (rp^{1/(p-1)})^{-1}$.

Theorem 3.5. *Let $x \in C\mathbb{Z}_p$, and let $s \in \mathbb{C}_p$ be such that $|s|_p < R_p = p^{(p-2)/(p-1)}$. Then there exists the convergent Laurent series expansion of $\zeta_{p,E}(s, x)$ such that*

$$\begin{aligned} \zeta_{p,E}(s, x) &= \int_{\mathbb{Z}_p} \langle x + a \rangle^{1-s} d\mu_{-1}(a) \\ &= \langle x \rangle^{1-s} \sum_{i=0}^{\infty} \binom{1-s}{i} E_i(0) \frac{1}{x^i}. \end{aligned}$$

Moreover, the function $\zeta_{p,E}(s, x)$ is a p -adic analytic on $|s|_p < R_p = p^{(p-2)/(p-1)}$.

Proof. It is immediate that

$$\langle x + a \rangle = \frac{x + a}{\omega_v(x + a)} = \frac{x}{\omega_v(x)} \left(1 + \frac{a}{x} \right)$$

since $\omega_v(1 + a/x) = 1$ for $x \in C\mathbb{Z}_p$. Hence

$$\langle x + a \rangle^{1-s} = \langle x \rangle^{1-s} \sum_{i=0}^{\infty} \binom{1-s}{i} a^i \frac{1}{x^i}.$$

And (3.3) gives $\langle x \rangle^{1-s}$ converges for $|s|_p < R_p$ (see [25, p. 54]). Next, if $f(a) = a^m$ in (2.1)-(2.3), where $m \geq 0$, then we have $\int_{\mathbb{Z}_p} (x+a+1)^m d\mu_{-1}(a) + \int_{\mathbb{Z}_p} (x+a)^m d\mu_{-1}(a) = 2x^m$, so that by (2.6) $E_m(x+1) + E_m(x) = 2x^m$. Therefore we get

$$\begin{aligned} 2 \sum_{a=0}^{\rho-1} (-1)^a a^m &= \sum_{a=0}^{\rho-1} (-1)^a (E_m(a+1) + E_m(a)) \\ &= \sum_{a=0}^{\rho-1} ((-1)^a E_m(a) - (-1)^{a+1} E_m(a+1)) \\ &= E_m(0) - (-1)^\rho E_m(\rho). \end{aligned}$$

If $\rho = p^N$ ($p > 2$) in this identity, then we have

$$\sum_{a=0}^{p^N-1} \langle x + a \rangle^{1-s} (-1)^a = \langle x \rangle^{1-s} \sum_{i=0}^{\infty} \binom{1-s}{i} \frac{1}{x^i} \frac{E_i(0) + E_i(p^N)}{2}.$$

From (2.5), we have

$$\begin{aligned} E_i(p^N) &= \sum_{k=0}^i \binom{i}{k} E_{i-k}(0) p^{kN} \\ &= E_i(0) + p^N \sum_{k=1}^i \binom{i}{k} E_{i-k}(0) p^{(k-1)N}. \end{aligned}$$

It follows that there exist $A_i(N) \in \mathbb{Z}_p$ such that $(E_i(0) + E_i(p^N))/2 = E_i(0) + p^N A_i(N)$ by $|E_i(0)|_p \leq 1$ (see (3.6) below) for $p > 2$, so that

$$\sum_{i=0}^{\infty} \binom{1-s}{i} \frac{E_i(0) + E_i(p^N)}{2x^i} = \sum_{i=0}^{\infty} \binom{1-s}{i} \frac{E_i(0)}{x^i} + p^N \sum_{i=0}^{\infty} \binom{1-s}{i} \frac{A_i(N)}{x^i}.$$

Since $|A_i(N)|_p \leq 1$ and $v_p((1 + 1/x)^{1-s}) = 0$ (see [3, p.216, Corollary 4.2.16]), it follows that

$$\begin{aligned} \left| \sum_{i=0}^{\infty} \binom{1-s}{i} \frac{1}{x^i} A_i(N) \right|_p &\leq \left| \sum_{i=0}^{\infty} \binom{1-s}{i} \frac{1}{x^i} \right|_p \\ &= \left| \left(1 + \frac{1}{x}\right)^{1-s} \right|_p \\ &= 1. \end{aligned}$$

Therefore we have

$$\lim_{N \rightarrow \infty} \sum_{a=0}^{p^N-1} \langle x+a \rangle^{1-s} (-1)^a = \langle x \rangle^{1-s} \sum_{i=0}^{\infty} \binom{1-s}{i} E_i(0) \frac{1}{x^i}.$$

This proves that there exists the convergent Laurent series expansion of $\zeta_{p,E}(s, x)$. Next we apply Lemma 3.4 to the series

$$(3.5) \quad \sum_{i=0}^{\infty} \binom{s}{i} E_i(0) \frac{1}{x^i}.$$

It is well known that Euler numbers E_0, E_1, E_2, \dots are integers and odd-numbered ones E_1, E_3, \dots are all zero (see [21]). From (2.9), we deduce that

$$E_i(0) = \sum_{k=0}^i \binom{i}{k} \frac{E_k}{2^k} \left(-\frac{1}{2}\right)^{i-k}.$$

Hence if $p > 2$, then we have

$$(3.6) \quad |E_i(0)|_p \leq \max_k \left\{ \left| \binom{i}{k} E_k \right|_p \right\} \leq 1$$

since $E_k \in \mathbb{Z}$. On the other hand, since $C\mathbb{Z}_p = \{x \in \mathbb{Q}_p \mid v_p(x) \leq -1\}$, we obtain $|x^{-1}|_p \leq p^{-1}$. Thus, observe that for odd p , $|E_i(0)x^{-i}|_p = |E_i(0)|_p |x^{-i}|_p \leq p^{-i}$ so that we can take $r = 1/p$ and $M = 1$ in Lemma 3.4. Therefore (3.5) converges in $D = \{s \in \mathbb{C}_p \mid |s|_p < R_p\} = \{s \in \mathbb{C}_p \mid |1-s|_p < R_p\}$. This proves that $\sum_{i=0}^{\infty} \binom{1-s}{i} E_i(0) x^{-i}$ is analytic in D . Since

$\langle x \rangle \equiv 1 \pmod{p\mathbb{Z}_p}$, we can assert that $\langle x \rangle^s$ is an analytic function in D and hence so is $\langle x \rangle^{1-s}$ (see [25, p. 54]). This completes the proof. \square

Corollary 3.6. *If $|s|_p < R_p$ and $x \in C\mathbb{Z}_p$, then*

$$\zeta_{p,E}(s, 1-x) = \zeta_{p,E}(s, x).$$

Proof. Using (2.12) and Theorem 3.5, we find

$$\begin{aligned} \zeta_{p,E}(s, 1-x) &= \langle -x \rangle^{1-s} \sum_{i=0}^{\infty} \binom{1-s}{i} E_i(1) (-1)^i \frac{1}{x^i} \\ &= \langle -x \rangle^{1-s} \sum_{i=0}^{\infty} \binom{1-s}{i} E_i(0) \frac{1}{x^i} \end{aligned}$$

whence the corollary follows from $\langle -x \rangle = \langle x \rangle$. \square

Lemma 3.7 ([4]). (1) *For $s \in \mathbb{C}_p$ such that $|s|_p < R_p$ and $x \in C\mathbb{Z}_p$,*

$$\frac{\partial}{\partial x} \langle x \rangle^{1-s} = (1-s) \langle x \rangle^{1-s} \frac{1}{x} = (1-s) \frac{1}{\langle x \rangle^s} \frac{1}{\omega_v(x)}.$$

(2) $\omega_v(x) = \omega_v(y)$ if y is sufficiently close to x .

Theorem 3.8. *Suppose that $x \in C\mathbb{Z}_p$. We have*

$$\zeta_{p,E}(1, x) = 1.$$

Proof. By Definition 3.2, we have

$$\zeta_{p,E}(1, x) = \int_{\mathbb{Z}_p} d\mu_{-1}(a) = \lim_{N \rightarrow \infty} \sum_{a=0}^{p^N-1} (-1)^a = 1.$$

\square

Theorem 3.9. *Suppose that $x \in C\mathbb{Z}_p$.*

(1) *For $m \in \mathbb{Z} \setminus \{0\}$,*

$$\zeta_{p,E}(1+m, x) = \omega_v^m(x) \int_{\mathbb{Z}_p} \frac{1}{(x+a)^m} d\mu_{-1}(a).$$

(2) *For $m > 0$,*

$$\zeta_{p,E}(1-m, x) = \frac{1}{\omega_v^m(x)} E_m(x) = \frac{1}{\omega_v^m(x)} \zeta_E(-m, x).$$

(3) *For $s \in \mathbb{C}_p$ such that $|s|_p < R_p$,*

$$\frac{\partial}{\partial x} \zeta_{p,E}(s, x) = \frac{1-s}{\omega_v(x)} \zeta_{p,E}(s+1, x).$$

Proof. By (3.4), we deduce that

$$\langle x+a \rangle = \frac{x+a}{\omega_v(x+a)} = \frac{1}{\omega_v(x)} (x+a)$$

since $\omega(1+a/x) = 1$ for $x \in C\mathbb{Z}_p$. Therefore, the proofs of (1) and (2) are easy from (3.2) and Definition 3.2. Now we will prove (3). It is immediate

that $(1-s-i)\binom{1-s}{i} = (1-s)\binom{-s}{i}$. Using (3.4), Definition 3.2, Lemma 3.7 and Theorem 3.5, we get the following identities

$$\begin{aligned} \frac{\partial}{\partial x} \zeta_{p,E}(s, x) &= \frac{\partial}{\partial x} \sum_{i=0}^{\infty} \binom{1-s}{i} E_i(0) \frac{1}{\langle x \rangle^{s+i-1}} \frac{1}{\omega_v(x)^i} \\ &= \sum_{i=0}^{\infty} (1-s-i) \binom{1-s}{i} E_i(0) \frac{1}{\langle x \rangle^{s+i}} \frac{1}{\omega_v(x)^{i+1}} \\ &= (1-s) \langle x \rangle^{1-s} \sum_{i=0}^{\infty} \binom{-s}{i} E_i(0) \frac{1}{x^{i+1}} \\ &= \frac{1-s}{\omega_v(x)} \zeta_{p,E}(s+1, x). \end{aligned}$$

Hence the proof of Theorem 3.9 is complete. \square

Theorem 3.10. (1) If $|s|_p < R_p$ and $x/u \in C\mathbb{Z}_p$, then

$$\zeta_{p,E}(s, x+u) = \langle x \rangle^{1-s} \sum_{i=0}^{\infty} \binom{1-s}{i} E_i(u) \frac{1}{x^i}.$$

(2) If $|s|_p < R_p$ and $x \in C\mathbb{Z}_p$, then we have the functional equation

$$\zeta_{p,E}(s, x+1) + \zeta_{p,E}(s, x) = \frac{2x}{\omega_v(x) \langle x \rangle^s}.$$

(3) If $|s|_p < R_p$, N is odd and $Nx \in C\mathbb{Z}_p$, then we have the distribution formula

$$\sum_{j=0}^{N-1} (-1)^j \zeta_{p,E} \left(s, x + \frac{j}{N} \right) = \zeta_{p,E}(s, Nx).$$

Proof. (1) Obviously, if $x/u \in C\mathbb{Z}_p$, we can write $\langle x+u \rangle = \langle x \rangle(1+u/x)$. One can check that

$$\begin{aligned} \zeta_{p,E}(s, x+u) &= \langle x+u \rangle^{1-s} \sum_{i=0}^{\infty} \binom{1-s}{i} E_i(0) \frac{1}{(x+u)^i} \\ &= \langle x \rangle^{1-s} \sum_{i=0}^{\infty} \binom{1-s}{i} E_i(0) \frac{1}{x^i} \left(1 + \frac{u}{x}\right)^{1-s-i} \\ &= \langle x \rangle^{1-s} \sum_{i=0}^{\infty} \binom{1-s}{i} E_i(0) \frac{1}{x^i} \sum_{j=0}^{\infty} \binom{1-s-i}{j} u^j \frac{1}{x^j} \\ &= \langle x \rangle^{1-s} \sum_{n=0}^{\infty} \sum_{i=0}^n \binom{1-s}{i} \binom{1-s-i}{n-i} E_i(0) u^{n-i} \frac{1}{x^n} \\ &= \langle x \rangle^{1-s} \sum_{n=0}^{\infty} \binom{1-s}{n} \frac{1}{x^n} \sum_{i=0}^n \binom{n}{i} E_i(0) u^{n-i} \\ &= \langle x \rangle^{1-s} \sum_{n=0}^{\infty} \binom{1-s}{n} E_n(u) \frac{1}{x^n} \end{aligned}$$

and so (1) is established. Next note that (1) trivially implies (2), by $E_n(x+1) + E_n(x) = 2x^n$ ($n \geq 0$) and $x = \omega_v(x)\langle x \rangle$. Now we prove (3). If N is odd, then one checks that

$$E_n(0) = N^n \sum_{j=0}^{N-1} (-1)^j E_n \left(\frac{j}{N} \right).$$

We conclude from (1) that

$$\begin{aligned} \sum_{j=0}^{N-1} (-1)^j \zeta_{p,E} \left(s, x + \frac{j}{N} \right) &= \sum_{j=0}^{N-1} (-1)^j \langle x \rangle^{1-s} \sum_{n=0}^{\infty} \binom{1-s}{n} \frac{1}{x^n} E_n \left(\frac{j}{N} \right) \\ &= \langle x \rangle^{1-s} \sum_{n=0}^{\infty} \binom{1-s}{n} \frac{1}{x^n} \frac{1}{N^n} E_n(0) = \zeta_{p,E}(s, Nx). \end{aligned}$$

This completes the proof. \square

We now prove a Raabe formula for $\zeta_{p,E}$ (cf. [5, Proposition 3.1]).

Theorem 3.11. *If $|s|_p < R_p$ and $x \in C\mathbb{Z}_p$, then*

$$\int_{\mathbb{Z}_p} \zeta_{p,E}(s, x+a) d\mu_{-1}(a) = 2 \left(1 + \frac{1}{x} \right) \zeta_{p,E}(s, x) - \frac{2}{x\langle x \rangle} \zeta_{p,E}(s-1, x).$$

Proof. Note that

$$\frac{1}{2} \left(\frac{2e^{2xt}}{e^t + 1} \right)^2 = \frac{2(1-2x)e^{2xt}}{e^t + 1} + \frac{d}{dt} \left(\frac{2e^{2xt}}{e^t + 1} \right)$$

(cf. [6]). From (2.5) and (2.6), we get the identity

$$(3.7) \quad \sum_{i=0}^m \binom{m}{i} E_i(x) E_{m-i}(x) = 2((1-2x)E_m(2x) + E_{m+1}(2x)), \quad m \geq 0.$$

Taking $x = 0$ in (3.7), we have

$$\begin{aligned} \int_{\mathbb{Z}_p} E_m(a) d\mu_{-1}(a) &= \sum_{i=0}^m \binom{m}{i} E_{m-i}(0) \int_{\mathbb{Z}_p} a^i d\mu_{-1}(a) \\ &= 2(E_m(0) + E_{m+1}(0)), \end{aligned}$$

since $E_m(a) = \sum_{i=0}^m \binom{m}{i} E_{m-i}(0) a^i$. From Theorem 3.10 (1), it follows that

$$\begin{aligned} (3.8) \quad & \int_{\mathbb{Z}_p} \zeta_{p,E}(s, x+a) d\mu_{-1}(a) \\ &= \langle x \rangle^{1-s} \sum_{i=0}^{\infty} \binom{1-s}{i} \frac{1}{x^i} \int_{\mathbb{Z}_p} E_i(a) d\mu_{-1}(a) \\ &= 2\langle x \rangle^{1-s} \sum_{i=0}^{\infty} \binom{1-s}{i} \frac{1}{x^i} (E_i(0) + E_{i+1}(0)) \\ &= 2\zeta_{p,E}(s, x) + 2\langle x \rangle^{1-s} \sum_{i=0}^{\infty} \binom{1-s}{i} \frac{1}{x^i} E_{i+1}(0). \end{aligned}$$

Using $\frac{\partial}{\partial x}$, we rewrite the function $\langle x \rangle^{1-s} \sum_{i=0}^{\infty} \binom{1-s}{i} x^{-i} E_{i+1}(0)$ in the right-hand side. If $|s|_p < R_p$ and $x \in C\mathbb{Z}_p$, we obtain

$$\begin{aligned} \frac{\partial}{\partial x} \zeta_{p,E}(s-1, x) &= (2-s) \left(\frac{1}{x} \zeta_{p,E}(s-1, x) \right. \\ &\quad \left. + \langle x \rangle^{2-s} \sum_{i=0}^{\infty} \binom{1-s}{i} \frac{1}{x^i} E_{i+1}(0) \right). \end{aligned}$$

Hence

$$\begin{aligned} \langle x \rangle^{1-s} \sum_{i=0}^{\infty} \binom{1-s}{i} \frac{1}{x^i} E_{i+1}(0) &= \frac{\langle x \rangle^{-1}}{2-s} \left(\frac{\partial}{\partial x} \zeta_{p,E}(s-1, x) \right. \\ &\quad \left. - \frac{2-s}{x} \zeta_{p,E}(s-1, x) \right). \end{aligned}$$

Substituting this identity in (3.8), it is easy to prove the desired result using Theorem 3.9 (3). \square

4. THE p -ADIC HURWITZ-TYPE EULER ZETA FUNCTIONS FOR $x \in \mathbb{Z}_p$

Let χ be a Dirichlet character modulo p^v for some v . We can extend the definition of χ to \mathbb{Z}_p as in [4, p.281]. In this section we define the p -adic Hurwitz-type Euler zeta functions for $x \in \mathbb{Z}_p$.

Definition 4.1. Let χ be a character modulo p^v with $v \geq 1$. If $x \in \mathbb{Z}_p$ and $s \in \mathbb{C}_p$ such that $|s|_p < R_p$ we define

$$\zeta_{p,E}(\chi, s, x) = \int_{\mathbb{Z}_p} \chi(x+a) \langle x+a \rangle^{1-s} d\mu_{-1}(a),$$

and we will simply write $\zeta_{p,E}(s, x)$ instead of $\zeta_{p,E}(\chi_0, s, x)$, where χ_0 is a trivial character modulo p^v .

We show that this definition makes sense. First we prove the following lemma on the change of variable for fermionic p -adic integrals which is an analogy of Lemma 11.2.3 in [4] on the change of variable for Volkenborn p -adic integrals.

Lemma 4.2. Let χ be a character modulo p^v , let f be a function defined for $v_p(x) < -v$ such that for fixed x the function $f(x+a)$ is in $UD(\mathbb{Z}_p)$, and set

$$g(x) = \int_{\mathbb{Z}_p} f(x+a) d\mu_{-1}(a).$$

Then for $x \in \mathbb{Z}_p$ we have

$$\sum_{j=0}^{p^v-1} \chi(x+j) g\left(\frac{x+j}{p^v}\right) (-1)^j = \int_{\mathbb{Z}_p} \chi(x+a) f\left(\frac{x+a}{p^v}\right) d\mu_{-1}(a).$$

Proof. By definition, we have

$$\begin{aligned}
& \sum_{j=0}^{p^v-1} \chi(x+j) g\left(\frac{x+j}{p^v}\right) (-1)^j \\
&= \lim_{r \rightarrow \infty} \sum_{j=0}^{p^v-1} \chi(x+j) (-1)^j \sum_{a=0}^{p^r-1} f\left(a + \frac{x+j}{p^v}\right) (-1)^a \\
&= \lim_{r \rightarrow \infty} \sum_{m=0}^{p^{v+r}-1} \chi(x+m) f\left(\frac{x+m}{p^v}\right) (-1)^m \\
&= \int_{\mathbb{Z}_p} \chi(x+a) f\left(\frac{x+a}{p^v}\right) d\mu_{-1}(a).
\end{aligned}$$

□

Corollary 4.3. *Definition 4.1 makes sense for $x \in \mathbb{Z}_p$ and $|s|_p < R_p$. More precisely, for any odd positive integer M such that $p^v | M$, we have*

$$\zeta_{p,E}(\chi, s, x) = \sum_{j=0}^{M-1} \chi(x+j) \zeta_{p,E}\left(s, \frac{x+j}{M}\right) (-1)^j.$$

Proof. First, we prove this Corollary for $M = p^v$. Applying the above lemma to $f(x) = \langle x \rangle^{1-s}$, we have

$$g(x) = \int_{\mathbb{Z}_p} \langle x+a \rangle^{1-s} \mu_{-1}(a) = \zeta_{p,E}(s, x),$$

thus

$$\begin{aligned}
& \sum_{j=0}^{p^v-1} \chi(x+j) \zeta_{p,E}\left(s, \frac{x+j}{p^v}\right) (-1)^j \\
(4.1) \quad &= \langle p^v \rangle^{s-1} \int_{\mathbb{Z}_p} \chi(x+a) \langle x+a \rangle^{1-s} d\mu_{-1}(a) \\
&= \zeta_{p,E}(\chi, s, x),
\end{aligned}$$

since $\langle p^v \rangle = 1$. For a general odd positive integer M , we write $M = Np^v$ and $j = p^v a + b$ with $0 \leq b < p^v$ and $0 \leq a < N$, so that

$$\begin{aligned}
& \sum_{j=0}^{M-1} \chi(x+j) \zeta_{p,E}\left(s, \frac{x+j}{M}\right) (-1)^j \\
&= \sum_{b=0}^{p^v-1} \chi(x+b) (-1)^b \sum_{a=0}^{N-1} \zeta_{p,E}\left(s, \frac{x+b}{Np^v} + \frac{a}{N}\right) (-1)^a \\
&= \sum_{b=0}^{p^v-1} \chi(x+b) \zeta_{p,E}\left(s, \frac{x+b}{p^v}\right) (-1)^b \\
&= \zeta_{p,E}(\chi, s, x),
\end{aligned}$$

using Theorem 3.10 (3) and (4.1).

□

Corollary 4.4. *Suppose that $x \in \mathbb{Z}_p$. We have*

$$\zeta_{p,E}(\chi, 1, x) = \sum_{j=0}^{p^v-1} \chi(x+j)(-1)^j.$$

Proof.

$$\zeta_{p,E}(\chi, 1, x) = \sum_{j=0}^{p^v-1} \chi(x+j) \zeta_{p,E} \left(1, \frac{x+j}{p^v} \right) (-1)^j = \sum_{j=0}^{p^v-1} \chi(x+j)(-1)^j$$

using Corollary 4.3 and Theorem 3.8. \square

Corollary 4.5. *Let χ be a character modulo p^v . Then for any $x \in \mathbb{Q}_p$ and any odd positive integer N such that $p^v | N$ and $Nx \in \mathbb{Z}_p$ we have*

$$\sum_{j=0}^{N-1} \chi(Nx+j) \zeta_{p,E} \left(s, x + \frac{j}{N} \right) (-1)^j = \zeta_{p,E}(\chi, s, Nx).$$

In particular,

$$\sum_{\substack{j=0 \\ p \nmid (Nx+j)}}^{N-1} \zeta_{p,E} \left(s, x + \frac{j}{N} \right) (-1)^j = \zeta_{p,E}(s, Nx),$$

where we recall that we have defined $\zeta_{p,E}(s, x) = \zeta_{p,E}(\chi_0, s, x)$ when $x \in \mathbb{Z}_p$.

Proof. Follows from Corollary 4.3 applied to $M = N$ and x is replaced by Nx . \square

We now begin to study the properties for the functions $\zeta_{p,E}(\chi, s, x)$ for $x \in \mathbb{Z}_p$.

Proposition 4.6. *If $x \in \mathbb{Z}_p$ we have*

$$\frac{\partial \zeta_{p,E}}{\partial x}(\chi, s, x) = (1-s) \zeta_{p,E}(\chi \omega^{-1}, s+1, x).$$

Proof. By Corollary 4.3 and Theorem 3.9 (3) we have

$$\begin{aligned} \frac{\partial \zeta_{p,E}}{\partial x}(\chi, s, x) &= (1-s) \sum_{j=0}^{p^v-1} \frac{\chi(x+j)}{p^v \omega_v \left(\frac{x+j}{p^v} \right)} \zeta_{p,E} \left(s+1, \frac{x+j}{p^v} \right) (-1)^j \\ &= (1-s) \sum_{j=0}^{p^v-1} \chi \omega^{-1}(x+j) \zeta_{p,E} \left(s+1, \frac{x+j}{p^v} \right) (-1)^j \\ &= (1-s) \zeta_{p,E}(\chi \omega^{-1}, s+1, x). \end{aligned}$$

\square

Proposition 4.7. *For fixed $x \in \mathbb{Z}_p$ the function $\zeta_{p,E}(\chi, s, x)$ is a p -adic analytic function on $|s|_p < R_p$.*

Proof. This is from Corollary 4.3 and Theorem 3.5. \square

Corollary 4.8. *We have*

$$\frac{\partial \zeta_{p,E}}{\partial x}(\chi\omega, 0, x) = \sum_{j=0}^{p^v-1} \chi(x+j)(-1)^j.$$

Proof. By Proposition 4.6, Proposition 4.7 and Corollary 4.4 we have

$$\begin{aligned} \frac{\partial \zeta_{p,E}}{\partial x}(\chi\omega, 0, x) &= \lim_{s \rightarrow 0} \frac{\partial \zeta_{p,E}}{\partial x}(\chi\omega, s, x) \\ &= \lim_{s \rightarrow 0} (1-s) \zeta_{p,E}(\chi, s+1, x) \\ &= \zeta_{p,E}(\chi, 1, x) \\ &= \sum_{j=0}^{p^v-1} \chi(x+j)(-1)^j. \end{aligned}$$

□

Proposition 4.9. *For $k \leq 1$ we have*

$$\zeta_{p,E}(\chi\omega^k, 1-k, x) = p^{vk} \sum_{j=0}^{p^v-1} \chi(x+j) E_k \left(\frac{x+j}{p^v} \right) (-1)^j.$$

Proof. By Corollary 4.3 and Theorem 3.9 (2) we have

$$\begin{aligned} \zeta_{p,E}(\chi\omega^k, 1-k, x) &= \sum_{j=0}^{p^v-1} \chi\omega^k(x+j) \zeta_{p,E} \left(1-k, \frac{x+j}{p^v} \right) (-1)^j \\ &= \sum_{j=0}^{p^v-1} \chi\omega^k(x+j) \omega_v \left(\frac{x+j}{p^v} \right)^{-k} E_k \left(\frac{x+j}{p^v} \right) (-1)^j \\ &= p^{vk} \sum_{j=0}^{p^v-1} \chi(x+j) E_k \left(\frac{x+j}{p^v} \right) (-1)^j. \end{aligned}$$

□

Theorem 4.10. *Let $x \in \mathbb{Z}_p$ and $|s|_p < R_p$.*

(1) *We have the functional equation*

$$\zeta_{p,E}(\chi, s, x+1) + \zeta_{p,E}(\chi, s, x) = 2\chi(x)\langle x \rangle^{1-s}.$$

(2) *We have the reflection formula*

$$\zeta_{p,E}(\chi, s, 1-x) = \chi(-1) \zeta_{p,E}(\chi, s, x).$$

(3) *Set*

$$\ell_{p,E}(\chi, s) = \zeta_{p,E}(\chi, s, 0).$$

If χ is an even character we have $\ell_{p,E}(\chi, s) = 0$, and more generally if n is a positive integer we have

$$(4.2) \quad \begin{aligned} & \zeta_{p,E}(\chi, s, n) \\ &= \chi(-1) \left(2 \sum_{j=1}^{n-1} \langle j-n \rangle^{1-s} \chi(j-n) (-1)^{j+1} + (-1)^{n+1} \ell_{p,E}(\chi, s) \right). \end{aligned}$$

(4) If $p \nmid N$ and N is odd we have the distribution formula

$$\sum_{i=0}^{N-1} \zeta_p \left(\chi, s, x + \frac{i}{N} \right) (-1)^i = \chi^{-1}(N) \zeta_{p,E}(\chi, s, Nx).$$

Remark 4.11. If χ is a character modulo p^v , then we have

$$(4.3) \quad \begin{aligned} \ell_{p,E}(\chi, s) &= \zeta_{p,E}(\chi, s, 0) \\ &= \int_{\mathbb{Z}_p} \langle a \rangle^{1-s} \chi(a) d\mu_{-1}(a) \\ &= \lim_{N \rightarrow \infty} \sum_{a=0}^{p^N-1} \langle a \rangle^{1-s} \chi(a) (-1)^a \\ &= \lim_{N \rightarrow \infty} \sum_{\substack{a=0 \\ p \nmid a}}^{p^N-1} \langle a \rangle^{1-s} \chi(a) (-1)^a, \end{aligned}$$

so the definition of $\ell_{p,E}(\chi, s)$ in (3) is the same as the first author's definition of $\ell_{p,E}(\chi, s)$ using Kubota-Leopoldt's approach (see [11, p. 6]).

Proof. (1) By definition we have

$$\begin{aligned} & \zeta_{p,E}(\chi, s, x+1) + \zeta_{p,E}(\chi, s, x) \\ &= \lim_{r \rightarrow \infty} \left(- \sum_{j=1}^{p^r} \chi(x+j) \langle x+j \rangle^{1-s} (-1)^j \right. \\ & \quad \left. + \sum_{j=0}^{p^r-1} \chi(x+j) \langle x+j \rangle^{1-s} (-1)^j \right) \\ &= \lim_{r \rightarrow \infty} (\chi(x) \langle x \rangle^{1-s} + \chi(x) \langle x+p^r \rangle^{1-s}) \\ &= 2\chi(x) \langle x \rangle^{1-s}, \end{aligned}$$

since $f(x) = \langle x \rangle^{1-s}$ is continuous for $|s|_p \leq R_p$ (see [25, p. 54]).

(2) By Corollary 4.3, Corollary 3.6 and setting $i = p^v - 1 - j$ we have

$$\begin{aligned}
& \zeta_{p,E}(\chi, s, 1-x) \\
&= \sum_{j=0}^{p^v-1} \chi(1-x+j) \zeta_{p,E} \left(s, \frac{1-x+j}{p^v} \right) (-1)^j \\
&= \sum_{i=0}^{p^v-1} \chi(p^v - i - x) \zeta_{p,E} \left(s, 1 - \frac{x+i}{p^v} \right) (-1)^i \\
&= \chi(-1) \sum_{i=0}^{p^v-1} \chi(x+i) \zeta_{p,E} \left(s, 1 - \frac{x+i}{p^v} \right) (-1)^i \\
&= \chi(-1) \sum_{i=0}^{p^v-1} \chi(x+i) \zeta_{p,E} \left(s, \frac{x+i}{p^v} \right) (-1)^i \\
&= \chi(-1) \zeta_{p,E}(\chi, s, x).
\end{aligned}$$

(3) From (1), we have $\zeta_{p,E}(\chi, s, 1) = -\zeta_{p,E}(\chi, s, 0)$ for any character. If $\chi(-1) = 1$, we have $\zeta_{p,E}(\chi, s, 1) = \zeta_{p,E}(\chi, s, 0)$ by (2), thus $\ell_{p,E}(\chi, s) = \zeta_{p,E}(\chi, s, 0) = 0$. For any positive integer n , we have

$$\begin{aligned}
& \zeta_{p,E}(\chi, s, n) \\
&= \chi(-1) \zeta_{p,E}(\chi, s, 1-n) \\
&= \chi(-1) (2 \langle 1-n \rangle^{1-s} \chi(1-n) - \zeta_{p,E}(\chi, s, 2-n)) \\
&= \chi(-1) \left(2 \sum_{j=1}^{n-1} \langle j-n \rangle^{1-s} \chi(j-n) (-1)^{j+1} + (-1)^{n+1} \ell_{p,E}(\chi, s) \right)
\end{aligned}$$

by using (1) and (2).

(4) By corollary 4.3 we have

$$\begin{aligned}
& \sum_{i=0}^{N-1} \zeta_{p,E} \left(\chi, s, x + \frac{i}{N} \right) (-1)^i \\
&= \sum_{i=0}^{N-1} \sum_{j=0}^{p^v-1} \chi \left(x + j + \frac{i}{N} \right) \zeta_{p,E} \left(s, \frac{x + j + i/N}{p^v} \right) (-1)^{j+i}.
\end{aligned}$$

Setting $a = Nj + i$ and using the fact that $p \nmid N$ and N is an odd positive integer, we have

$$\begin{aligned}
& \sum_{i=0}^{N-1} \zeta_{p,E} \left(\chi, s, x + \frac{i}{N} \right) (-1)^i \\
&= \chi^{-1}(N) \sum_{a=0}^{Np^v-1} \chi(Nx + a) \zeta_{p,E} \left(s, \frac{Nx + a}{Np^v} \right) (-1)^a \\
&= \chi^{-1}(N) \zeta_{p,E}(\chi, s, Nx)
\end{aligned}$$

using Corollary 4.3. □

Next we prove the Raabe formula of the p -adic Hurwitz-type Euler zeta functions for $x \in \mathbb{Z}_p$.

Theorem 4.12. *Let χ be a character modulo p^v . For $|s|_p < R_p$ and $x \in \mathbb{Z}_p$ we have*

$$\int_{\mathbb{Z}_p} \zeta_{p,E}(\chi, s, x+a) d\mu_{-1}(a) = 2(1-x)\zeta_{p,E}(\chi, s, x) + 2\zeta_{p,E}(\chi\omega, s-1, x).$$

Proof. From the definition and Corollary 4.3, we have

$$\begin{aligned} & \int_{\mathbb{Z}_p} \zeta_{p,E}(\chi, s, x+a) d\mu_{-1}(a) \\ &= \lim_{r \rightarrow \infty} \sum_{i=0}^{p^r-1} \zeta_{p,E}(\chi, s, x+i) (-1)^i \\ &= \lim_{r \rightarrow \infty} \sum_{i=0}^{p^r-1} \left(\sum_{j=0}^{p^r-1} \chi(x+i+j) \zeta_{p,E} \left(\chi, s, \frac{x+i+j}{p^r} \right) (-1)^j \right) (-1)^i. \end{aligned}$$

Setting $n \equiv i+j \pmod{p^r}$ in other words the unique $n \equiv i+j \pmod{p^r}$ such that $0 \leq n < p^r$. We can have only $i+j = n$ or $i+j = n+p^r$ and the number of pairs (i, j) such that $i+j = n$ is equal to $n+1$, which the number of pairs such that $i+j = n+p^r$ is equal to $p^r - (n+1)$. Let

$$S(r) = \sum_{i=0}^{p^r-1} \left(\sum_{j=0}^{p^r-1} \chi(x+i+j) \zeta_{p,E} \left(\chi, s, \frac{x+i+j}{p^r} \right) (-1)^j \right) (-1)^i.$$

We have

$$\int_{\mathbb{Z}_p} \zeta_{p,E}(\chi, s, x+a) d\mu_{-1}(a) = \lim_{r \rightarrow \infty} S(r)$$

and

$$\begin{aligned} (4.4) \quad S(r) &= \sum_{n=0}^{p^r-1} \chi(x+n) \left((n+1) \zeta_{p,E} \left(s, \frac{x+n}{p^r} \right) (-1)^n \right. \\ &\quad \left. + (p^r - (n+1)) \zeta_{p,E} \left(s, \frac{x+n}{p^r} + 1 \right) (-1)^{n+1} \right) \\ &= \sum_{n=0}^{p^r-1} (-1)^n \chi(x+n) \left((n+1) \zeta_{p,E} \left(s, \frac{x+n}{p^r} \right) \right. \\ &\quad \left. - (p^r - (n+1)) \zeta_{p,E} \left(s, \frac{x+n}{p^r} + 1 \right) \right). \end{aligned}$$

By Theorem 3.10 (2), we have

$$\begin{aligned}
 (4.5) \quad & \zeta_{p,E} \left(s, \frac{x+n}{p^r} + 1 \right) + \zeta_{p,E} \left(s, \frac{x+n}{p^r} \right) \\
 &= 2 \frac{\frac{x+n}{p^r}}{\omega_v \left(\frac{x+n}{p^r} \right) \langle x+n \rangle^s} \\
 &= 2 \frac{x+n}{\omega_v(x+n) \langle x+n \rangle^s} \\
 &= 2 \frac{\langle x+n \rangle}{\langle x+n \rangle^s} \\
 &= 2 \langle x+n \rangle^{1-s}.
 \end{aligned}$$

By (4.5) we have

$$(4.6) \quad \zeta_{p,E} \left(s, \frac{x+n}{p^r} + 1 \right) = 2 \langle x+n \rangle^{1-s} - \zeta_{p,E} \left(s, \frac{x+n}{p^r} \right).$$

Substitute (4.6) into (4.4) we have

$$\begin{aligned}
 (4.7) \quad S(r) &= \sum_{n=0}^{p^r-1} (-1)^n \chi(x+n) \left((n+1) \zeta_{p,E} \left(s, \frac{x+n}{p^r} \right) \right. \\
 &\quad \left. - (p^r - (n+1)) \left(2 \langle x+n \rangle^{1-s} - \zeta_{p,E} \left(s, \frac{x+n}{p^r} \right) \right) \right) \\
 &= \sum_{n=0}^{p^r-1} (-1)^n \chi(x+n) \left((n+1) \zeta_{p,E} \left(s, \frac{x+n}{p^r} \right) \right. \\
 &\quad \left. - (p^r - (n+1)) 2 \langle x+n \rangle^{1-s} \right. \\
 &\quad \left. + (p^r - (n+1)) \zeta_{p,E} \left(s, \frac{x+n}{p^r} \right) \right) \\
 &= \sum_{n=0}^{p^r-1} (-1)^n \chi(x+n) (n+1) \zeta_{p,E} \left(s, \frac{x+n}{p^r} \right) \\
 &\quad - \sum_{n=0}^{p^r-1} \chi(x+n) (p^r - (n+1)) 2 \langle x+n \rangle^{1-s} \\
 &\quad + \sum_{n=0}^{p^r-1} (-1)^n \chi(x+n) (p^r - (n+1)) \zeta_{p,E} \left(s, \frac{x+n}{p^r} \right) \\
 &= \sum_{n=0}^{p^r-1} (-1)^n \chi(x+n) p^r \zeta_{p,E} \left(s, \frac{x+n}{p^r} \right) \\
 &\quad - \sum_{n=0}^{p^r-1} (-1)^n \chi(x+n) (p^r - (n+1)) 2 \langle x+n \rangle^{1-s}.
 \end{aligned}$$

By corollary 4.3 we have

$$(4.8) \quad \begin{aligned} S(r) &= p^r \zeta_{p,E}(\chi, s, x) \\ &\quad - \sum_{n=0}^{p^r-1} (-1)^n \chi(x+n) (p^r - (n+1)) 2 \langle x+n \rangle^{1-s}. \end{aligned}$$

From above equality we have

$$(4.9) \quad \begin{aligned} &\lim_{r \rightarrow \infty} S(r) \\ &= -2 \lim_{r \rightarrow \infty} \sum_{n=0}^{p^r-1} (-1)^n \chi(x+n) (p^r - (n+1)) \langle x+n \rangle^{1-s}. \end{aligned}$$

Since

$$(4.10) \quad \begin{aligned} &\sum_{n=0}^{p^r-1} (-1)^n \chi(x+n) (p^r - (n+1)) \langle x+n \rangle^{1-s} \\ &= \sum_{n=0}^{p^r-1} (-1)^n \chi(x+n) (p^r + x - 1 - (x+n)) \langle x+n \rangle^{1-s} \\ &= p^r \sum_{n=0}^{p^r-1} (-1)^n \chi(x+n) \langle x+n \rangle^{1-s} \\ &\quad + \sum_{n=0}^{p^r-1} (-1)^n (x-1) \langle x+n \rangle^{1-s} \chi(x+n) \\ &\quad - \sum_{n=0}^{p^r-1} (-1)^n \chi(x+n) (x+n) \langle x+n \rangle^{1-s} \\ &= p^r \sum_{n=0}^{p^r-1} (-1)^n \chi(x+n) \langle x+n \rangle^{1-s} \\ &\quad + \sum_{n=0}^{p^r-1} (-1)^n (x-1) \langle x+n \rangle^{1-s} \chi(x+n) \\ &\quad - \sum_{n=0}^{p^r-1} (-1)^n \chi(x+n) \omega_v(x+n) \langle x+n \rangle \langle x+n \rangle^{1-s} \\ &= p^r \sum_{n=0}^{p^r-1} (-1)^n \chi(x+n) \langle x+n \rangle^{1-s} \\ &\quad + \sum_{n=0}^{p^r-1} (-1)^n (x-1) \langle x+n \rangle^{1-s} \chi(x+n) \\ &\quad - \sum_{n=0}^{p^r-1} (-1)^n \chi \omega(x+n) \langle x+n \rangle^{1-(s-1)} \end{aligned}$$

and

$$\begin{aligned} \lim_{r \rightarrow \infty} p^r \sum_{n=0}^{p^r-1} (-1)^n \chi(x+n) \langle x+n \rangle^{1-s} &= 0, \\ \lim_{r \rightarrow \infty} \sum_{n=0}^{p^r-1} (-1)^n (x-1) \langle x+n \rangle^{1-s} \chi(x+n) &= (x-1) \zeta_{p,E}(\chi, s, x), \\ \lim_{r \rightarrow \infty} - \sum_{n=0}^{p^r-1} (-1)^n \chi \omega(x+n) \langle x+n \rangle^{1-(s-1)} &= -\zeta_{p,E}(\chi \omega, s-1, x), \end{aligned}$$

we have

$$(4.11) \quad \lim_{r \rightarrow \infty} - \sum_{n=0}^{p^r-1} (-1)^n \chi \omega(x+n) \langle x+n \rangle^{1-(s-1)} = -\zeta_{p,E}(\chi \omega, s-1, x),$$

and

$$\begin{aligned} (4.12) \quad \int_{\mathbb{Z}_p} \zeta_{p,E}(\chi, s, x+a) d\mu_{-1}(a) &= \lim_{r \rightarrow \infty} S(r) \\ &= -2((x-1) \zeta_{p,E}(\chi, s, x) - \zeta_{p,E}(\chi \omega, s-1, x)) \\ &= 2(1-x) \zeta_{p,E}(\chi, s, x) + 2 \zeta_{p,E}(\chi \omega, s-1, x). \end{aligned}$$

□

Next we prove the following power series expansion in x of $\zeta_{p,E}(\chi, s, x)$.

Proposition 4.13. *Let χ be a character modulo p^v for some $v \geq 1$. For $x \in p^v \mathbb{Z}_p$ we have the power series expansion*

$$\zeta_{p,E}(\chi, s, x) = \sum_{k=0}^{\infty} \binom{1-s}{k} \ell_{p,E}(\chi \omega^{-k}, s+k) x^k,$$

where $\ell_{p,E}(\chi, s) = \zeta_{p,E}(\chi, s, 0)$ is the p -adic Euler ℓ -function.

Proof. By definition, we have

$$\begin{aligned} \zeta_{p,E}(\chi, s, x) &= \int_{\mathbb{Z}_p} \chi(x+a) \langle x+a \rangle^{1-s} d\mu_{-1}(a) \\ &= \int_{\mathbb{Z}_p^\times} \chi(a) \langle a \rangle^{1-s} \langle 1+x/a \rangle^{1-s} d\mu_{-1}(a) \\ &= \int_{\mathbb{Z}_p^\times} \sum_{k=0}^{\infty} \binom{1-s}{k} \chi \omega^{-k}(a) \langle a \rangle^{1-s-k} x^k d\mu_{-1}(a) \\ &= \sum_{k=0}^{\infty} \binom{1-s}{k} x^k \int_{\mathbb{Z}_p} \chi \omega^{-k}(a) \langle a \rangle^{1-s-k} d\mu_{-1}(a) \\ &= \sum_{k=0}^{\infty} \binom{1-s}{k} \zeta_{p,E}(\chi \omega^{-k}, s+k, 0) x^k \\ &= \sum_{k=0}^{\infty} \binom{1-s}{k} \ell_{p,E}(\chi \omega^{-k}, s+k) x^k, \end{aligned}$$

where $\mathbb{Z}_p^\times = \mathbb{Z}_p \setminus p\mathbb{Z}_p$. □

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DEPARTMENT OF MATHEMATICS, KOREA ADVANCED INSTITUTE OF SCIENCE AND TECHNOLOGY (KAIST), 373-1 GUSEONG-DONG, YUSEONG-GU, DAEJEON 305-701, SOUTH KOREA

E-mail address: minsookim@kaist.ac.kr

DEPARTMENT OF MATHEMATICS, KOREA ADVANCED INSTITUTE OF SCIENCE AND TECHNOLOGY (KAIST), 373-1 GUSEONG-DONG, YUSEONG-GU, DAEJEON 305-701, SOUTH KOREA

E-mail address: husu@kaist.ac.kr, suhu1982@gmail.com